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Applied and Computational Mechanics 2 (2008) 157–166

Applied and Computational Mechanics

Comparison study of the computational methods for eigenvalues IFE analysis

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Received 28 August 2008; received in revised form 6 October 2008

Abstract

The goal of the paper is to present a non-traditional computational tool for structural analysis with uncertainties in material, geometric and load parameters. Uncertainties are introduced as bounded possible values — intervals. The main objective has been to propose algorithms for interval modal and spectral computations on FEM models suggested by authors and their comparison with Monte Carlo.

Keywords: uncertain parameter, interval arithmetic, Monte Carlo, optimization

1. Introduction

In the last decade, there has been an increased interest in the modeling and analysis of engineering systems under uncertainties [11]. Computational mechanics, for example, encounters uncertainties in geometric, material and load parameters as well as in the model itself and in the analysis procedure too. For that reason, the responses, such as displacements, stresses, natural frequency, or other dynamic characteristics, will usually show some degree of uncertainty [10, 11]. It means that the obtained result using one specific value as the most significant value for an uncertain parameter cannot be considered as representative for the whole spectrum of possible results.

It is generally known that probabilistic modeling and statistical analysis are well established for modeling of mechanical systems with uncertainties. In addition, a number of non-probabilistic computational techniques have been proposed, e.g. fuzzy set theory [1, 11], interval approach [2, 3, 5, 7, 8, 9], imprecise probabilities [4, 9] etc. The growing interest in these approaches originated from a criticism of the credibility of probabilistic approach when input dates are insufficient. It is argued that the new non-probabilistic treatments could be more appropriate in the modeling of the vagueness.

2. Interval analysis

Interval arithmetic was developed by Moore [3] while studying the propagation and control of truncation and rounding off the error, using floating point arithmetic on a digital computer. Moore was able to generalize this work into the arithmetic independence of machine considerations. In this approach, an uncertain number is represented by an interval of real numbers. The interval numbers derived from the experimental data or expert knowledge can then take into

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account the uncertainties in the model parameters, model inputs etc. By this technique, the complete information about the uncertainties in the model may be included and one can demonstrate how these uncertainties are processed by the calculation procedure in MATLAB [6]. Considering uncertain parameters in interval form, we will realize comparison study of the eigenvalue problem (INTLAB function — *verifyeig*).

The alternative avenue of the interval arithmetic is to use the Monte Carlo technique (MC) [1, 4]. With the advent of recent computational facilities, this method becomes attractive. The results are determined from the series of numerical analyses (approximately 1 000–10 000 iterations). It is recommended to generate the random values with the uniform distribution.

The comparison has been realized with the usage of midpoint residual vector $\mathbf{r}_{\mathrm{Midpoint}}$ and radius residual vector $\mathbf{r}_{\mathrm{Residual}}$ expressed in %, e.g.

$$\begin{aligned} \mathbf{r}_{\mathrm{Midpoint}} &= \left|\frac{\mathrm{mid}\,(\mathbf{y}_{\mathrm{Intlab}})-\mathrm{mid}\,(\mathbf{y}_{\mathrm{MC}})}{\mathrm{mid}\,(\mathbf{y}_{\mathrm{Intlab}})}\right| \cdot 100\,\%, \\ \mathbf{r}_{\mathrm{Radius}} &= \left|\frac{\mathrm{rad}\,(\mathbf{y}_{\mathrm{Intlab}})-\mathrm{rad}\,(\mathbf{y}_{\mathrm{MC}})}{\mathrm{rad}\,(\mathbf{y}_{\mathrm{Intlab}})}\right| \cdot 100\,\%. \end{aligned} \tag{1}$$

During the solving of the particular tasks in the engineering practice using the interval arithmetic application on the solution of numerical mathematics and mechanical problems, the problem known as the overestimate effect is encountered. Its elimination is possible only in the case of meeting the specific assumptions, mainly related to the time efficiency of the computing procedures. Now, we will try to analyze some solution approaches already used or proposed by the authors. We will consider the following methods:

- Monte Carlo method (MC) as a "reference method",
- method of a solution evaluation in marginal values of interval parameters — infimum and supremum (COM1),
- method of a solution evaluation for all marginal values of interval parameters — all combinations of infimum and supremum (COM2),
- method of infimum and supremum searching using some optimizing technique application (OPT),
- direct application of the interval arithmetic using INTLAB — MATLAB's toolbox (INTL), [7].

Testing example

Let's compare the proposed interval computational methods during solving of the following eigenvalues problem

A) with a "small" signification (about $\pm 2.5\,\%$) of the parameters uncertainty, e.g.

$$\left(\begin{bmatrix} \langle 19\,400\;19\,500\rangle & -\langle 9\,400\;9\,450\rangle \\ -\langle 9\,400\;9\,450\rangle & \langle 9\,400\;9\,450\rangle \end{bmatrix} - \lambda_i \cdot \begin{bmatrix} \langle 20\;20.1\rangle & 0 \\ 0 & \langle 18\;18.1\rangle \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

B) and with a "larger" signification (about $\pm 5\,\%$) of the parameters uncertainty, e.g.

$$\left(\begin{bmatrix} \langle 19\,400\;21\,400\rangle & -\langle 9\,400\;10\,400\rangle \\ -\langle 9\,400\;10\,400\rangle & \langle 9\,400\;10\,400\rangle \end{bmatrix} - \lambda_i \cdot \begin{bmatrix} \langle 20\;25\rangle & 0 \\ 0 & \langle 18\;22\rangle \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Inf-sup results have been compiled into Table 1. Comparison of the proposed methods using (1) is presented in Table 1 and 2. They are shown errors in midpoints and in radiuses. Presented results suggest effectiveness of the COM2 and OPT approaches.

Table 1. Results obtained by the proposed numerical methods

		MC	COM1	COM2	OPT	INT
A	λ_1	$\langle 201\ 203\rangle$	$\langle 202.3\ 202.5\rangle$	$\langle 201\ 203\rangle$	$\langle 201\ 203\rangle$	$\langle 199\ 206\rangle$
	λ_2	$\langle 1\,284\ 1\,296\rangle$	$\langle 1\,289\ 1\,290\rangle$	$\langle 1\,283\ 1\,296\rangle$	$\langle 1\,283\ 1\,296\rangle$	$\langle 1\,268\ 1\,312\rangle$
B	λ_1	$\langle 167\ 220\rangle$	$\langle 181\ 202\rangle$	$\langle 164\ 223\rangle$	$\langle 164\ 223\rangle$	$\langle 95\ 287\rangle$
	λ_2	$\langle 1\,059\ 1\,415\rangle$	$\langle 1\,147\ 1\,290\rangle$	$\langle 1\,039\ 1\,425\rangle$	$\langle 1\,038\ 1\,425\rangle$	$\langle 600\ 1\,827\rangle$

Table 2. Errors in midpoints

λ_i		MC	COM1		COM2		OPT		INT	
		Refer. midpoint	Midp.	Error [%]	Midp.	Error [%]	Midp.	Error [%]	Midp.	Error [%]
A	λ_1	202	202.4	0.2	202	0	202	0	202.5	0.25
	λ_2	1 290	1289.5	0.04	1 289.5	0.04	1289.5	0.04	1 290	0
B	λ_1	193	191.5	1.03	193.5	0	193.5	0	191	1.29
	λ_2	1 237	1 218.5	1.5	1 232	0.4	1 231.5	0.44	1 213.5	1.9

Table 3. Errors in radiuses

λ_i		MC	COM1		COM2		OPT		INT	
		Refer. radius	Radius	Error [%]	Radius	Error [%]	Radius	Error [%]	Radius	Error [%]
A	λ_1	1	0.1	90	1	0	1	0	3.5	250
	λ_2	6	0.5	91.67	6.5	8.33	6.5	8.33	22	266.67
B	λ_1	26.5	10.5	60.38	29.5	11.32	29.5	11.32	96	262.26
	λ_2	178	71.5	59.83	193	8.43	193.5	8.71	613.5	244.66

3. Interval Eigenvalues Finite Elements Analysis (IFEA)

The finite element method (FEM) [2, 9, 11] is generally a very popular tool for a structural analysis. The ability to predict the response of a structure under static or dynamic loads is not only of a great scientific value, it is also very useful from an economical point of view. A reliable FE analysis could reduce the need for prototype production and therefore significantly reduce the associated design validation cost.

It is sometimes very difficult to define a reliable FE model for realistic mechanical structures when a number of its physical properties is uncertain. Particularly, in the case of FE analysis, the mechanical properties of the used materials are very hard to predict, and therefore an important source of uncertainty. Reliable validation can only be based on an analysis which takes into account all uncertainties that could cause this variability.

According to the character of the uncertainty, we can define a structural uncertainty (geometrical and material parameters) and uncertainty in load (external forces, etc.). The structural uncertainty parameters are usually written into vector $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and the interval modal and spectral FE analysis may be formulated as follows

$$\lfloor \mathbf{K}(\mathbf{x}) - \lambda_j \cdot \mathbf{M}(\mathbf{x}) \rfloor \cdot \mathbf{v}_j = \mathbf{0} \quad \text{or} \quad \left(\lfloor \underline{\mathbf{K}}, \overline{\mathbf{K}} \rfloor - [\lambda_j, \lambda_j] \cdot \lfloor \underline{\mathbf{M}}, \overline{\mathbf{M}} \rfloor \right) \cdot \left\lfloor \underline{\mathbf{v}_j}, \overline{\mathbf{v}_j} \right\rfloor = \mathbf{0}, \tag{2}$$

where $\underline{\lambda_j}, \overline{\lambda_j}$ and $\underline{\mathbf{v}_j}, \overline{\mathbf{v}_j}$ are the j-th eigenvalue with corresponding eigenvector, $\underline{\mathbf{K}}$, $\overline{\mathbf{K}}$ and $\underline{\mathbf{M}}$, $\overline{\mathbf{M}}$ are the infimum and supremum of the mass and stiffness matrices. The application of the classic interval arithmetic for FE analysis is very limited. Its "overestimation" grows with the problem size (the dimension of the system matrices) and has not a physical foundation in the reality. Therefore, it is efficient to apply the previous numerical methods. Application of the Monte Carlo method in IFEA may be realized in the following computational steps:

1. step: generation of the random matrix (uniform distribution)

$$\mathbf{X}_{MC} = [\mathbf{x}_1, \ldots, \mathbf{x}_m], \qquad (m \approx 5\,000 \div 100\,000),$$

2. step: solution of

$$\boldsymbol{\lambda}_{j\_MC} \rightarrow \lfloor \mathbf{K}(\mathbf{x}_j) - \boldsymbol{\lambda}_{j\_MC} \cdot \mathbf{M}(\mathbf{x}_j) \rfloor \cdot \mathbf{V}_j = \mathbf{0} \quad \text{for } j = 1, ldots, m,$$

3. step:
– infimum calculation of the i-th eigenvalue $\quad \underline{\underline{\lambda_i}} = \inf\left(i^{\text{th}} \text{ row of } \lambda_{MC}\right)$,
– supremum calculation of the i-th eigenvalue $\quad \overline{\overline{\lambda_i}} = \sup\left(i^{\text{th}} \text{ row of } \lambda_{MC}\right)$.

In the case of the COM1, the numerical approach implementation to IFEA is following:
– only infimum calculation $\quad \underline{\underline{\boldsymbol{\lambda}}} \rightarrow [\mathbf{K}(\underline{\mathbf{x}}) - \underline{\boldsymbol{\lambda}} \cdot \mathbf{M}(\underline{\mathbf{x}})] \cdot \underline{\mathbf{V}} = \mathbf{0}$,
– only supremum calculation $\quad \overline{\overline{\boldsymbol{\lambda}}} \rightarrow \left[\mathbf{K}(\overline{\mathbf{x}}) - \overline{\lambda} \cdot \mathbf{M}(\overline{\mathbf{x}})\right] \cdot \overline{\mathbf{V}} = \mathbf{0}$.

COM1 doesn't give the correct results every time, especially in the case of the large matrices $\mathbf{K}$ and M. We can obtain more proper results using COM2. Its computational process for IFEA includes these steps:

1. step: calculation of realizations matrix $\mathbf{X}_2$, i.e. 2^n inf-sup combinations,

$$\mathbf{X}_{COM2} = [\mathbf{x}_1, \ldots, \mathbf{x}_m], \quad (m = 2^n), \qquad n \text{ — number of uncertain system parameters},$$

2. step: solution of

$$\boldsymbol{\lambda}_{j\_COM2} \rightarrow \lfloor \mathbf{K}(\mathbf{x}_j) - \boldsymbol{\lambda}_{j\_COM2} \cdot \mathbf{M}(\mathbf{x}_j) \rfloor \cdot \mathbf{V}_j = \mathbf{0} \quad \text{for} \quad j = 1, \ldots, m,$$

3. step:
– infimum calculation of the i-th eigenvalue $\quad \underline{\underline{\lambda_i}} = \inf\left(i^{\text{th}} \text{ row of } \boldsymbol{\lambda}_{COM2}\right)$,
– supremum calculation of the i-th eigenvalue $\quad \overline{\overline{\lambda_i}} = \sup\left(i^{\text{th}} \text{ row of } \boldsymbol{\lambda}_{COM2}\right)$.

Generally, the infimum or supremum are not found only in the boundary points (COM1, COM2) but also in the inner domain of the solution set. To find the inf-sup solution using the approach OPT means to solve the optimizing problem described as follows:
– infimum calculation of the i-th eigenvalue

$$\underline{\lambda_i}(\mathbf{x}_{OPT}) \rightarrow \text{ minimize value of } \lambda_i \text{ for eq.: } [\mathbf{K}(\mathbf{x}) - \boldsymbol{\lambda}_i \cdot \mathbf{M}(\mathbf{x})] \cdot \mathbf{v}_i = \mathbf{0},$$

– supremum calculation of the i-th eigenvalue

$$\overline{\lambda_i}(\mathbf{x}_{OPT}) \rightarrow \text{ maximize value of } \lambda_i \text{ for eq.: } [\mathbf{K}(\mathbf{x}) - \lambda_i \cdot \mathbf{M}(\mathbf{x})] \cdot \mathbf{v}_i = \mathbf{0}.$$

It should be noted that it is possible to realize the searching process by a comparison optimizing method (e.g. Nelder-Mead simplex algorithm) or by using genetic algorithm as a robust tool of the global optimization.

4. Interval eigenvalue solution of the fourdrinier structure

For the following research purposes on the interval finite element model computing, the frame structure shown on the Fig. 1 was analyzed. The certain material parameters are defined as follows element mass density $\rho = 2\,700\,\text{kg} \cdot \text{m}^{-3}$ and Young's modulus $E = 2 \cdot 10^{11}$ Pa.

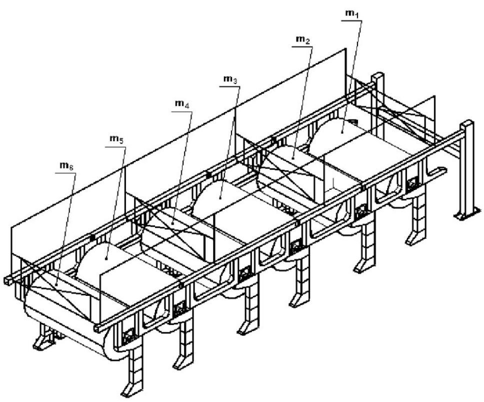

Fig. 1. Analyzed fourdrinier structure

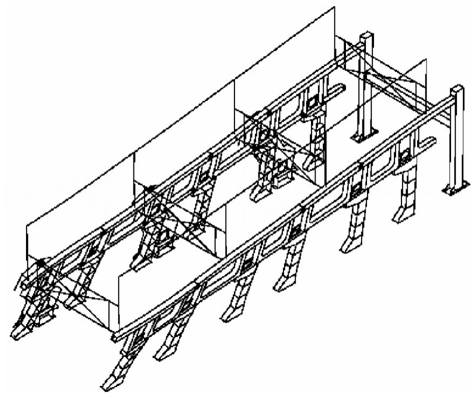

Fig. 2. Modal shape for natural freq. $f_1 = 3.69\,\text{Hz}$

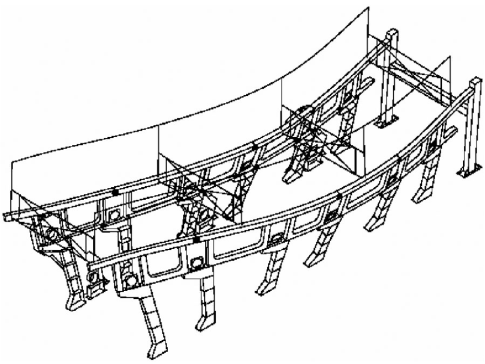

Fig. 3. Modal shape for natural freq. $f_2 = 4.17\,\text{Hz}$

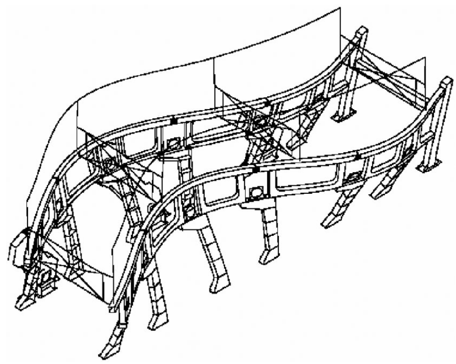

Fig. 4. Modal shape for natural freq. $f_3 = 5.56\,\text{Hz}$

The purpose of this study is to compare the efficiency and exactness of the proposed methods MC, COM1, COM2 and OPT. The results of the MC analysis are considered as the reference values and are used for the construction of the solution map. The interval modal and spectral analysis of the identical structure was realized assuming the following uncertain model parameters (mass of cylinders, Fig. 1):

$$\mathbf{x} = [m_1, m_2, m_3, m_4, m_5, m_6]\,, \text{ where } m_{1\div 6} \in \langle 0, 2\,200 \rangle \text{ [kg]}\,. \tag{3}$$

The solution will now consider only the analyses of the first three interval natural frequencies f_1, f_2 and f_3 (See Figs. 2, 3, 4). In the case of MC method, 2000 random inputs have been generated, evaluated and properly processed to inf-sup solutions

$$\inf = \min\left(f_i(\mathbf{x})\right), \qquad \sup = \max\left(f_i(\mathbf{x})\right). \tag{4}$$

The interval solution results are summarized in the Tabs. 4, 5, 6. The graphical representation for the all computational approaches is shown on Fig. 5. This figure presents map of the input data set for each of the method. The "iterative" map with the infimum and supremum of the first two natural frequencies obtained using the suggested methods is shown on the Fig. 6. By reason of the effectiveness of the COM2 and OPT methods it has been realized short comparison their iteration history. Figs. 7 and 8 present iteration history of the cylinder masses and Figs. 9 and 10 maps the iteration process of the natural frequencies f_1, f_2.

Table 4. Results of the natural frequencies

Freq. no.	**MC**	**COM1**	**OPT**	**COM2**
$\mathbf{f}_1$ [Hz]	$\langle 3.1947\ 3.6262 \rangle$	$\langle 3.1720\ 3.6780 \rangle$	$\langle 3.1717\ 3.6720 \rangle$	$\langle 3.1717\ 3.6781 \rangle$
$\mathbf{f}_2$ [Hz]	$\langle 3.6226\ 4.0816 \rangle$	$\langle 3.5940\ 4.1730 \rangle$	$\langle 3.5936\ 4.0854 \rangle$	$\langle 3.5936\ 4.1735 \rangle$
$\mathbf{f}_3$ [Hz]	$\langle 4.8166\ 5.4976 \rangle$	$\langle 4.7690\ 5.5610 \rangle$	$\langle 4.7685\ 5.2445 \rangle$	$\langle 4.7685\ 5.5615 \rangle$

Table 5. Errors in midpoints of the natural frequencies

Freq. no.	**MC**	**COM1**		**OPT**		**COM2**	
	Reference midpoint	Midpoint	Error [%]	Midpoint	Error [%]	Midpoint	Error [%]
f_1 [Hz]	3.410 45	3.425 0	0.427	3.421 85	0.334	3.424 9	0.424
f_2 [Hz]	3.852 1	3.883 5	0.815	3.839 5	0.327	3.883 55	0.816
f_3 [Hz]	5.157 1	5.165	0.153	5.006 5	2.92	5.165	0.153

Table 6. Errors in radiuses of the natural frequencies

Freq. no.	**MC**	**COM1**		**OPT**		**COM2**	
	Reference radius	Radius	Error [%]	Radius	Error [%]	Radius	Error [%]
f_1 [Hz]	0.215 75	0.253	17.265	0.250 15	15.944	0.253 2	17.358
f_2 [Hz]	0.229 5	0.289 5	26.144	0.245 9	7.146	0.289 95	26.340
f_3 [Hz]	0.340 5	0.396	16.300	0.238	30.103	0.396 5	16.446

On the basis of the experiences obtained from the interval estimations of the FEM models spectral properties it is possible to conclude:

- the appropriateness of OPT algorithm application, which mainly due to the simplicity of the criteria function for the infimum or supremum analysis gives excellent results, in some cases even better than MC method,
- the previous fact relates to the application of the genetic searching algorithms, of which the biggest advantage is their universality and particularly searching for global extremes,
- the inappropriateness of the COM1 method is demonstrated again because it shows a considerable deflection against the other methods,
- the COM2 method has a limited use but it can be suitable in combination with the OPT method because of "locating" of the solution map corner solutions.

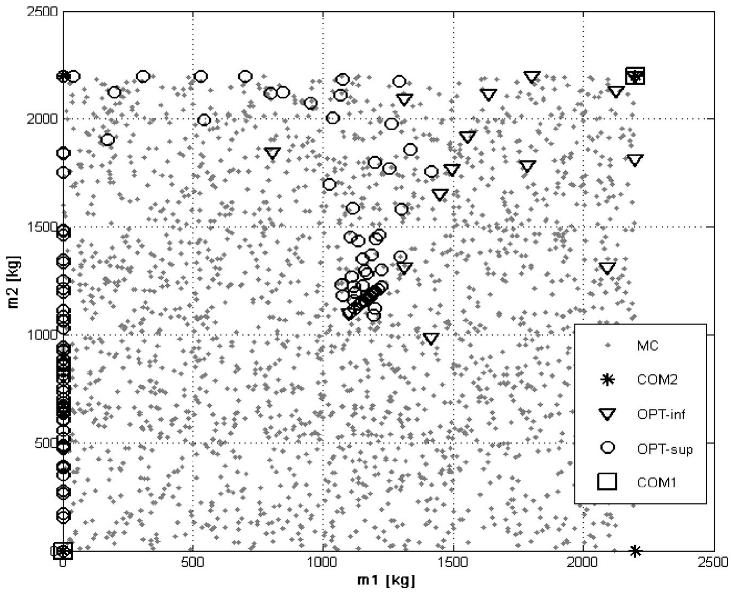

Fig. 5. Mapping of the generated input data for applied methods

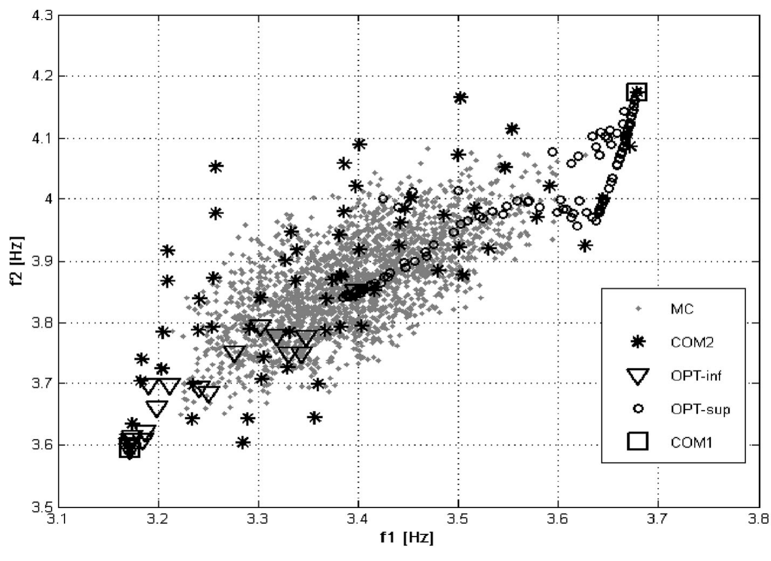

Fig. 6. Mapping of the solution history for 1st and 2nd natural frequencies

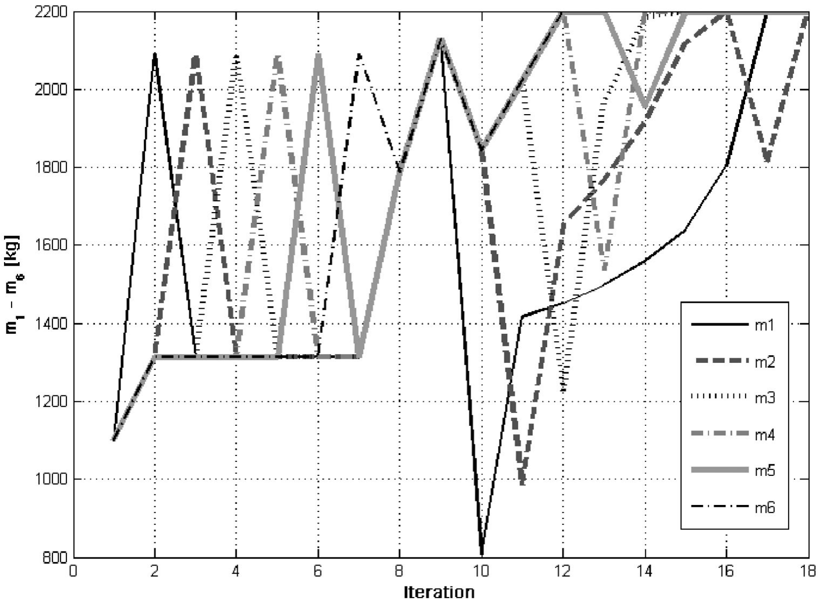

Fig. 7. History of the iteration process for cylinder masses — COM2

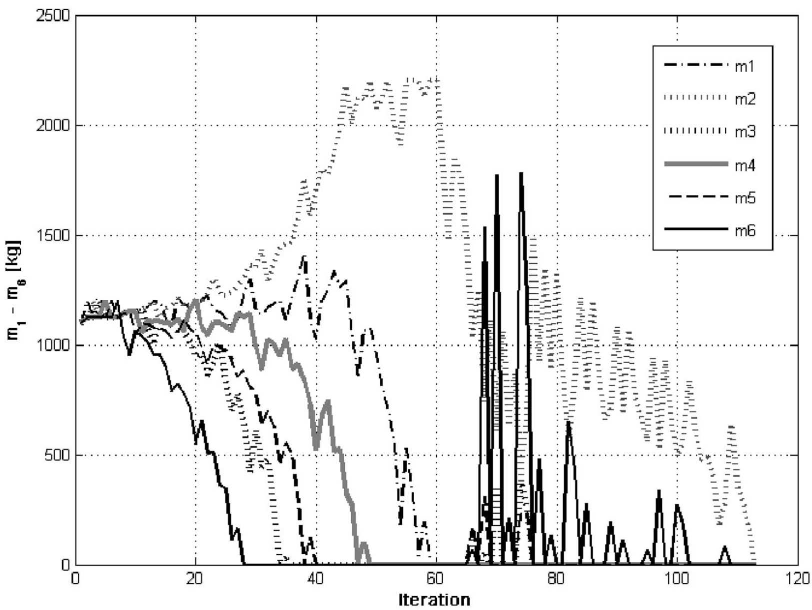

Fig. 8. History of the iteration process for cylinder masses — OPT

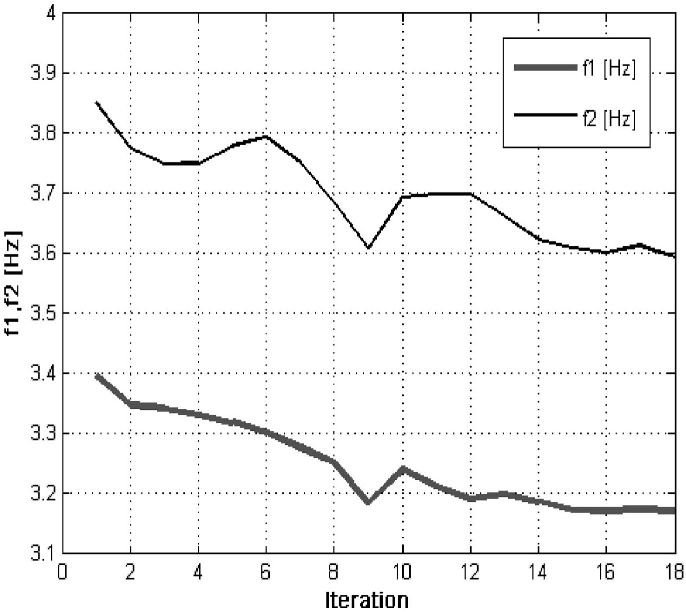

Fig. 9. History of the iteration process for natural frequencies f_1, f_2 — COM2

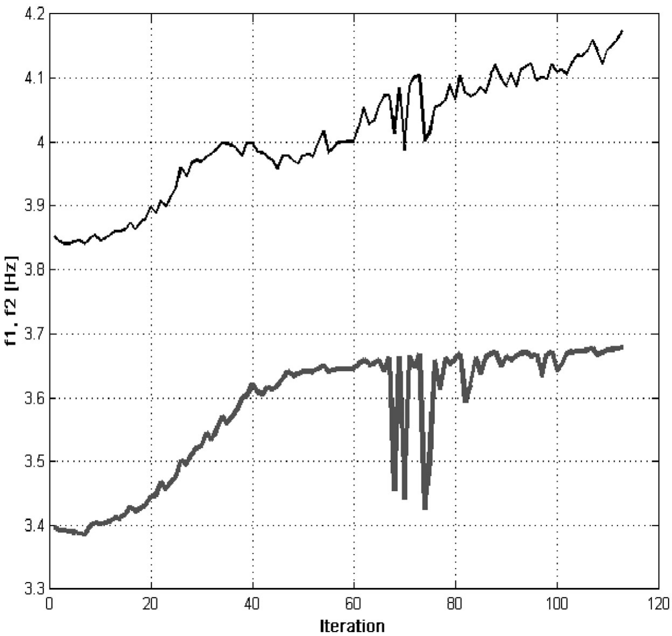

Fig. 10. History of the iteration process for natural frequencies f_1, f_2 — OPT

5. Conclusion

The paper discusses the possibility of the interval arithmetic application in a modal and spectral FE analysis. The interval arithmetic provides a new possibility of the quality and reliability appraisal of analyzed objects. Due to this numerical approach, we can analyze mechanical, technological, service and economic properties of the investigated structures more authentically.

In our paper we have investigated possibilities of the modal and spectral solution of a fourdrinier structure with an interval weight of the cylinders (according to condensed water). The centre of our interest has been mainly to compare the suggested numerical algorithms and their efficiency evaluation.

Acknowledgements

This work has been supported by VEGA grant No. 1/3168/06 and VEGA grant No. 1/4099/07.

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